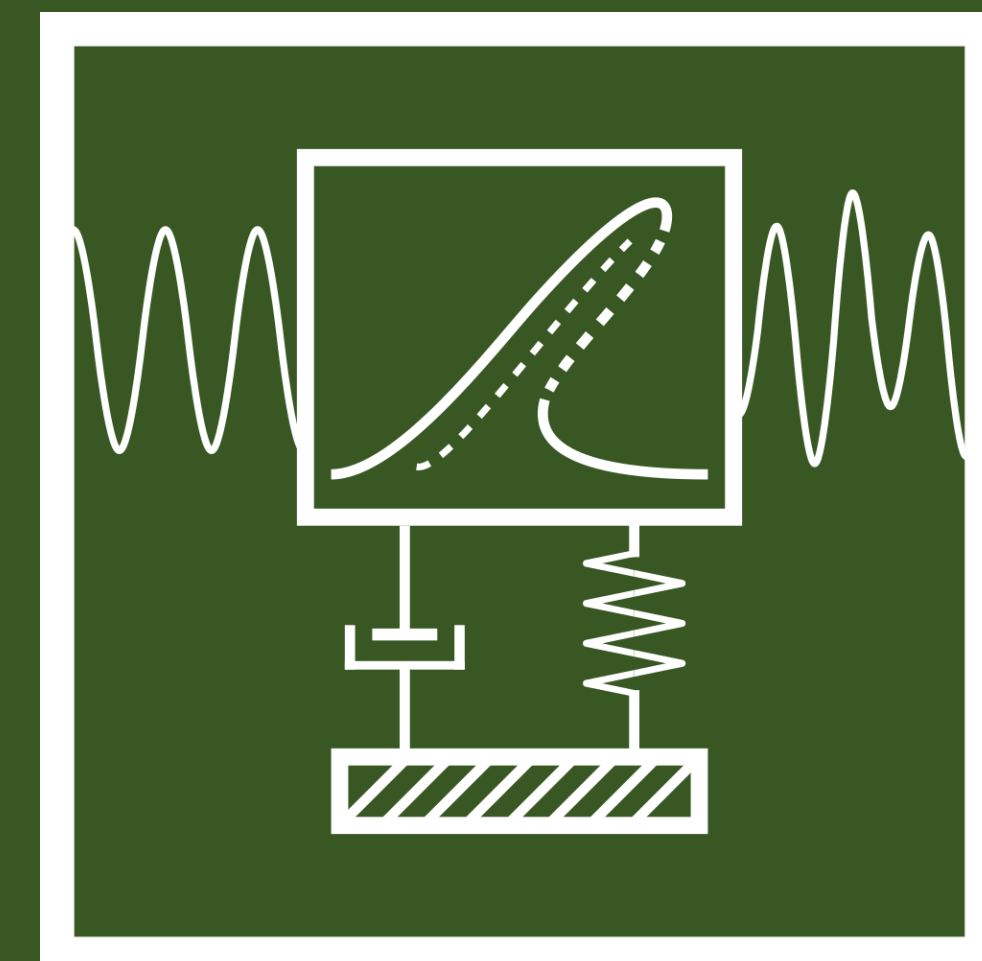


Approximation of fractional differential equations via integer order ordinary differential equations

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A Introduction and Aims

This work addresses the inherent ambiguity in separating a fractional system's dynamics to perform structural analysis of a given fractional differential equation (FDE). Instead of relying on a single, manually derived representation, this study treats the integer-order component $G(y(x))$ as a flexible, parameterizable, and optimizable polynomial.

Primary objectives:

Optimize Model Architecture: To introduce a global optimization framework (using Particle Swarm Optimization) that searches the space of possible ODE architectures by parameterizing the polynomial coefficients of $G(y(x))$.

Minimize Dual Objectives: To identify an optimal, low-degree polynomial structure that simultaneously minimizes the solution error (compared to a high-fidelity reference) and the magnitude of the truncated residual series $\Psi_M(t^n)$.

Uncover Structural Insights: To provide a novel analytical tool that reveals deeper fundamental insights into the complex interplay between local, integer-order dynamics and non-local memory effects within fractional systems.

01 FDE to ODE Transformation

Initial FDE

$$\begin{aligned} (\mathbf{D}^{\frac{1}{n}})^k y(x) &= F(y(x)), \quad 1 \leq k < n; \\ y(0) &= v_0, (\mathbf{D}^{\frac{1}{n}})^j y(x) \Big|_{x=0} = v_j, \quad j = 1, \dots, k-1. \end{aligned}$$

Equivalent First-Order ODE

Apply substitution $t = x^{\frac{1}{n}}$ (with $\hat{y}(t) = y(t^n)$) to reduce the transformed FDE to an equivalent first-order ODE:

$$\frac{d\hat{y}}{dt} = nt^{n-1} \left(G(\hat{y}(t)) + \Psi(t^n) \right) + \sum_{j=1}^{n-1} \frac{jv_j}{\Gamma\left(1 + \frac{j}{n}\right)} t^{j-1}$$

Approximate ODE for Optimization

Truncate $\Psi(t^n)$ for numerical computation ($\Psi(t^n) \approx \Psi_M(t^n)$), yielding the final approximate ODE:

$$\frac{d\tilde{y}}{dt} = nt^{n-1} \left(G(\tilde{y}(t)) + \Psi_M(t^n) \right) + \sum_{j=1}^{n-1} \frac{jv_j}{\Gamma\left(1 + \frac{j}{n}\right)} t^{j-1}$$

Figure 1. Optimized residual fractional series $\Psi_M(t^3)$ for the Riccati FDE

$$(\mathbf{D}^{\frac{1}{n}})^k y(x) = a_0 + a_1 y + a_2 y^2$$

for polynomial degrees $N_G = 1, \dots, 5$. The visualization confirms that while the $N_G = 1$ model yields the smallest residual, the optimal $N_G = 3$ model remains well-behaved. The large and oscillatory residuals for the higher-degree models ($N_G \geq 4$) provide a visual explanation for their suboptimal performance.

02 Optimization Algorithm

Parametrize $G(y)$

Define $G(y)$ as a polynomial whose coefficients $\mathbf{b}$ are the target for optimization:

$$G(y; \mathbf{b}) = \sum_{p=0}^{N_G} b_p y^p$$

Establish Normalization Baseline

Define a baseline architecture $\mathbf{b}_{\text{baseline}}$, using the original FDE's structure. Calculate its performance metrics to serve as reference values for normalization:

$$\text{RMSE}_{\text{ref}} = \text{RMSE}(\mathbf{b}_{\text{baseline}}); \quad \|\Psi_M\|_{\text{ref}} = \|\Psi_M(\mathbf{b}_{\text{baseline}})\|_2$$

Iterative Optimization Loop (Particle Swarm Optimization)

- Generate candidate and solve:** for a candidate vector $\mathbf{b}$, construct and solve the corresponding approximate ODE to the solution $\tilde{y}(t; \mathbf{b})$.
- Evaluate fitness:** calculate the solution accuracy (RMSE) and model simplicity ($\|\Psi_M\|_2$) and combine them into a single normalized objective function:

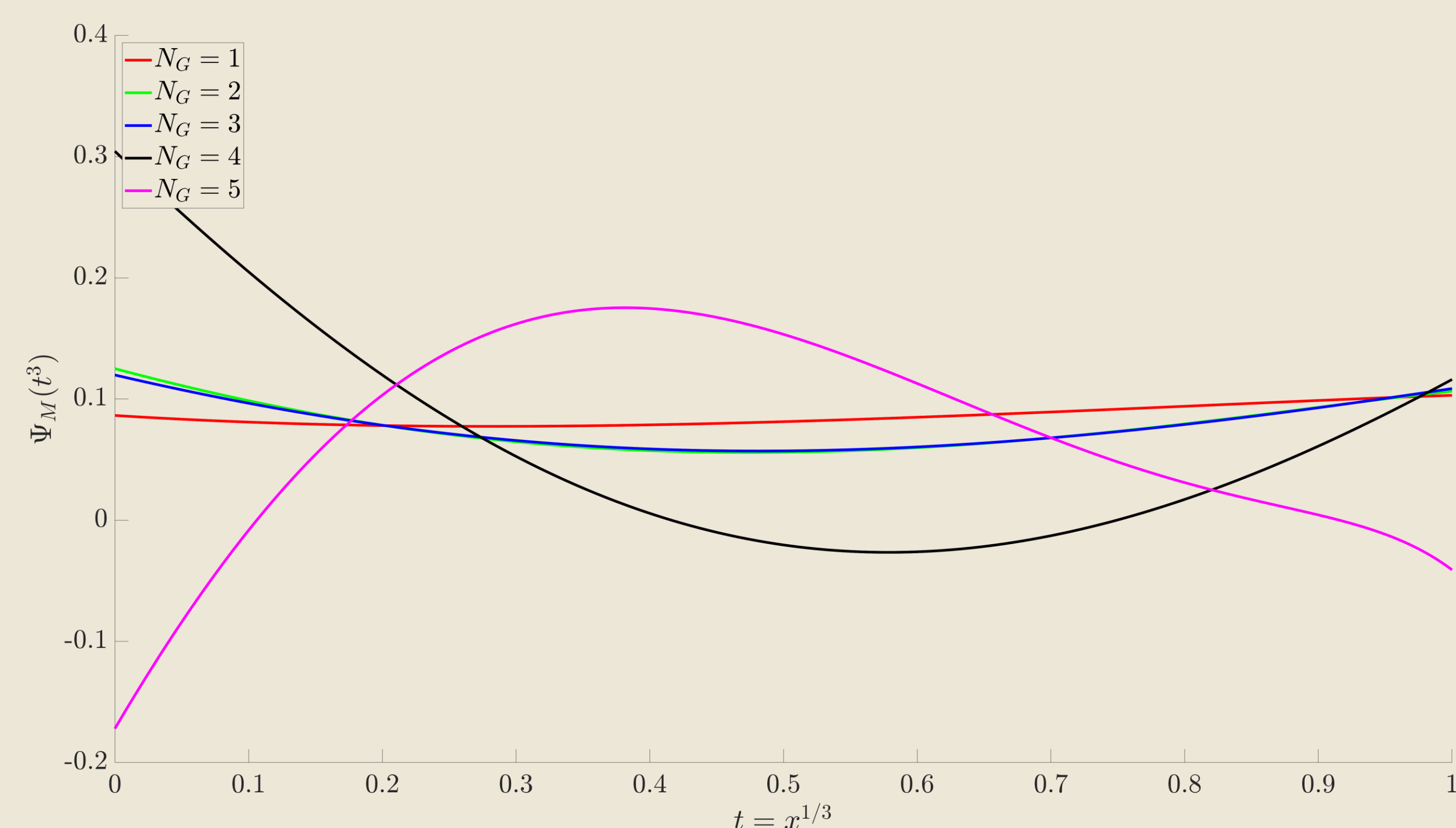
$$\text{Minimize}_{\mathbf{b}} \quad T_{\text{norm}}(\mathbf{b}) = \kappa_1 \frac{\text{RMSE}(\mathbf{b})}{\text{RMSE}_{\text{ref}}} + \kappa_2 \frac{\|\Psi_M(\mathbf{b})\|_2}{\|\Psi_M\|_{\text{ref}}}$$

Optimal Architecture

The optimization yields the coefficient vector $\mathbf{b}^*$, which defines the optimal architecture $G(y; \mathbf{b}^*)$.

03 Conclusions

By balancing solution accuracy with structural simplicity, the proposed optimization framework identifies parsimonious integer-order representations of fractional differential equations, providing a novel analytical tool to uncover the fundamental interplay between local dynamics and non-local memory effects.



References

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