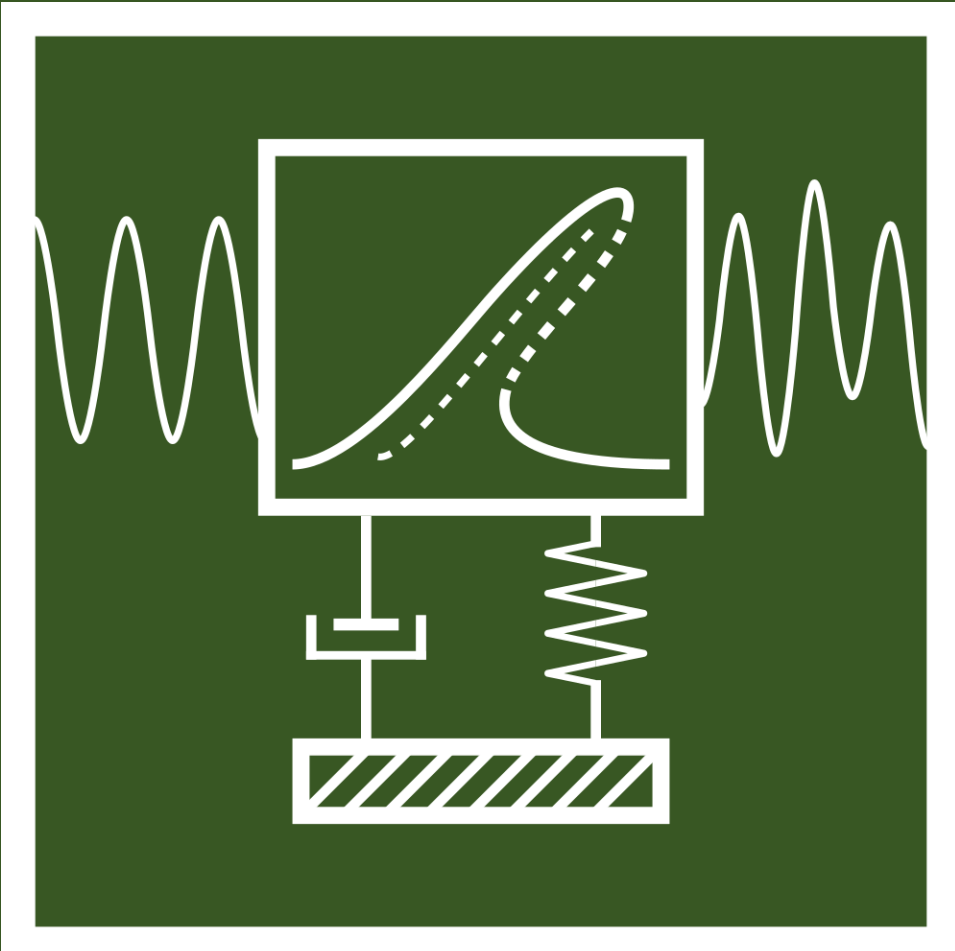


# From Time Series to Reconstructed Dynamics: Non-Uniform Embedding and its Applications



Kristina Poškuvienė

Department of Mathematical Modelling, Kaunas University of Technology, Kaunas, Lithuania

Corresponding author e-mail: kristina.poskuviene@ktu.lt

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## Abstract

Traditional uniform time-delay embedding uses a common delay between reconstruction coordinates. For systems with multiple temporal scales, this restriction can hide informative dynamical structure. Non-uniform embedding allows individual delays to be selected and therefore adapts the observation scales to the signal. This poster summarizes a geometric strategy for selecting informative delays and illustrates how reconstructed dynamics have been used for synchronization analysis, fault diagnosis, structural damage identification, recurrence-based visualization, and group physiological dynamics.

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## Methods

### Why State-Space Embedding?

Delay-coordinate embedding reconstructs the underlying dynamics of a system from a scalar time series  $\{x_1, x_2, x_3, \dots, x_N\}$  by representing successive system states in a multidimensional space Fig. 1.

### Uniform embedding

The conventional reconstruction uses equally spaced coordinates:

$$Y_i = [x_i, x_{i+\tau}, \dots, x_{i+(m-1)\tau}]$$

where  $\tau$  is the time delay;  $m$  is the embedding dimension. The embedding dimension is commonly estimated using the False Nearest Neighbours (FNN) method, whereas the delay is frequently selected using autocorrelation, average mutual information (AMI), or geometric criteria.

### Why Non-Uniform Embedding?

Complex dynamics may involve multiple temporal scales; therefore, a single delay may produce redundant or insufficiently informative coordinates. Non-uniform embedding allows different delays:

$$Y_i = [x_i, x_{i+\tau_1}, x_{i+\tau_1+\tau_2}, \dots, x_{i+\Delta}], \quad \Delta = \sum_{k=1}^{m-1} \tau_k$$

where  $i = 1, 2, \dots, N - \Delta$ .

The delays become adaptive observation scales selected to unfold the reconstructed dynamics.

### Geometry-Based Non-Uniform Embedding

Following the geometry-based approach introduced in [1], the delays are selected by maximizing the geometric spread of the reconstructed trajectory:

$$(\tau_1^*, \tau_2^*, \dots, \tau_{m-1}^*) = \operatorname{argmax} F(\tau_1, \tau_2, \dots, \tau_{m-1}),$$
$$F(\tau_1, \tau_2, \dots, \tau_{m-1}) = \frac{1}{(N - \Delta)\sqrt{m}} \sum_{i=1}^{N-\Delta} \sqrt{x_i^2 + x_{i+\tau_1}^2 + x_{i+\tau_1+\tau_2}^2 + \dots + x_{i+\Delta}^2}.$$

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## Results

- **Extended delay space.** Non-uniform embedding allows delays to be selected independently, beyond the constraint  $\tau_1 = \tau_2$ .
- **Non-uniform optima.** For both systems, the geometric optimum lies outside the uniform diagonal, with  $F_{NU} > F_U$  (Fig. 2).
- **Multiple temporal scales.** Unequally spaced coordinates provide a more flexible representation of complex dynamics.
- **Broad applicability.** Reconstructed dynamics support forecasting, complexity and recurrence analysis, synchronization, and group-dynamics studies (Fig. 3).

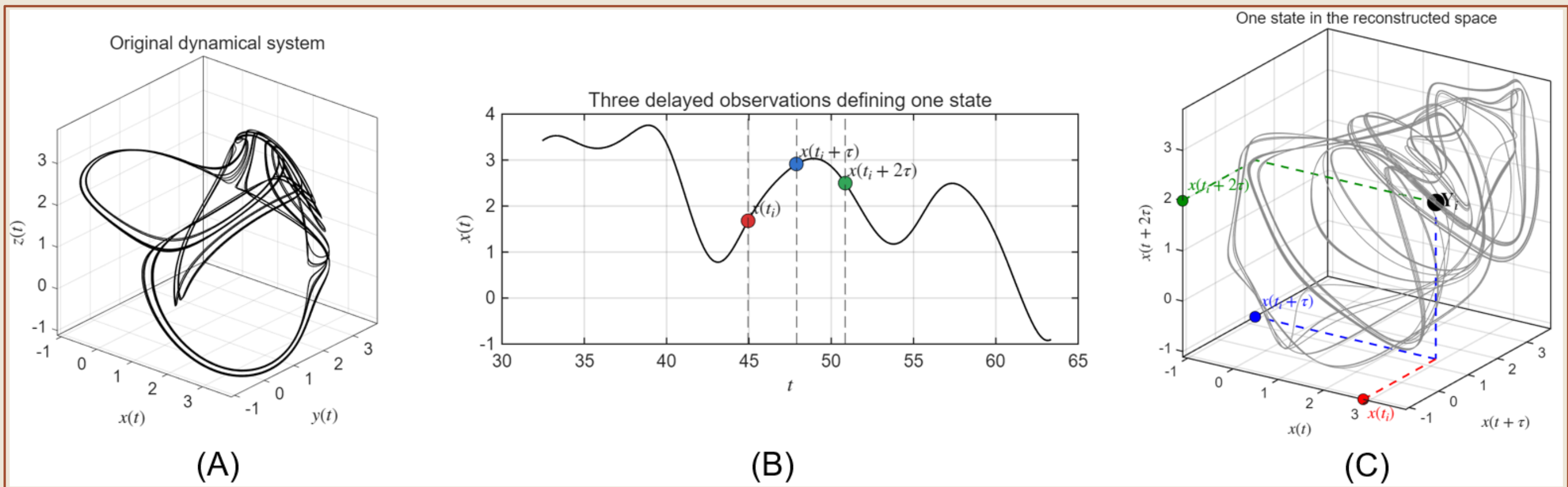


Fig 1. Reconstruction of the underlying dynamics from a single observed time series.

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## Discussion

Non-uniform embedding removes the restriction of a single characteristic delay and allows individual coordinates to represent different temporal scales. The examples in Fig. 2 show that the optimum of the geometric criterion may lie outside the uniform constraint ( $\tau_1 = \tau_2$ ), providing a more flexible representation of the observed dynamics.

This flexibility is particularly relevant for complex and multiscale signals, where a single delay may not capture all informative temporal relationships. As illustrated in Fig. 3, the reconstructed state-space representation can subsequently support different analysis tasks, including forecasting, complexity assessment, recurrence analysis, synchronization, and group-dynamics analysis.

The main trade-off is an increased parameter-search space; therefore, an appropriate optimization criterion is essential for systematic delay selection.

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## Conclusions

- Non-uniform embedding generalizes classical delay-coordinate reconstruction by allowing independently selected temporal coordinates.
- Geometry-based optimization provides a systematic way to identify informative delay combinations without imposing the uniform-delay constraint.
- The resulting reconstructed dynamics provide a flexible representation for a broad range of nonlinear time-series analysis tasks.

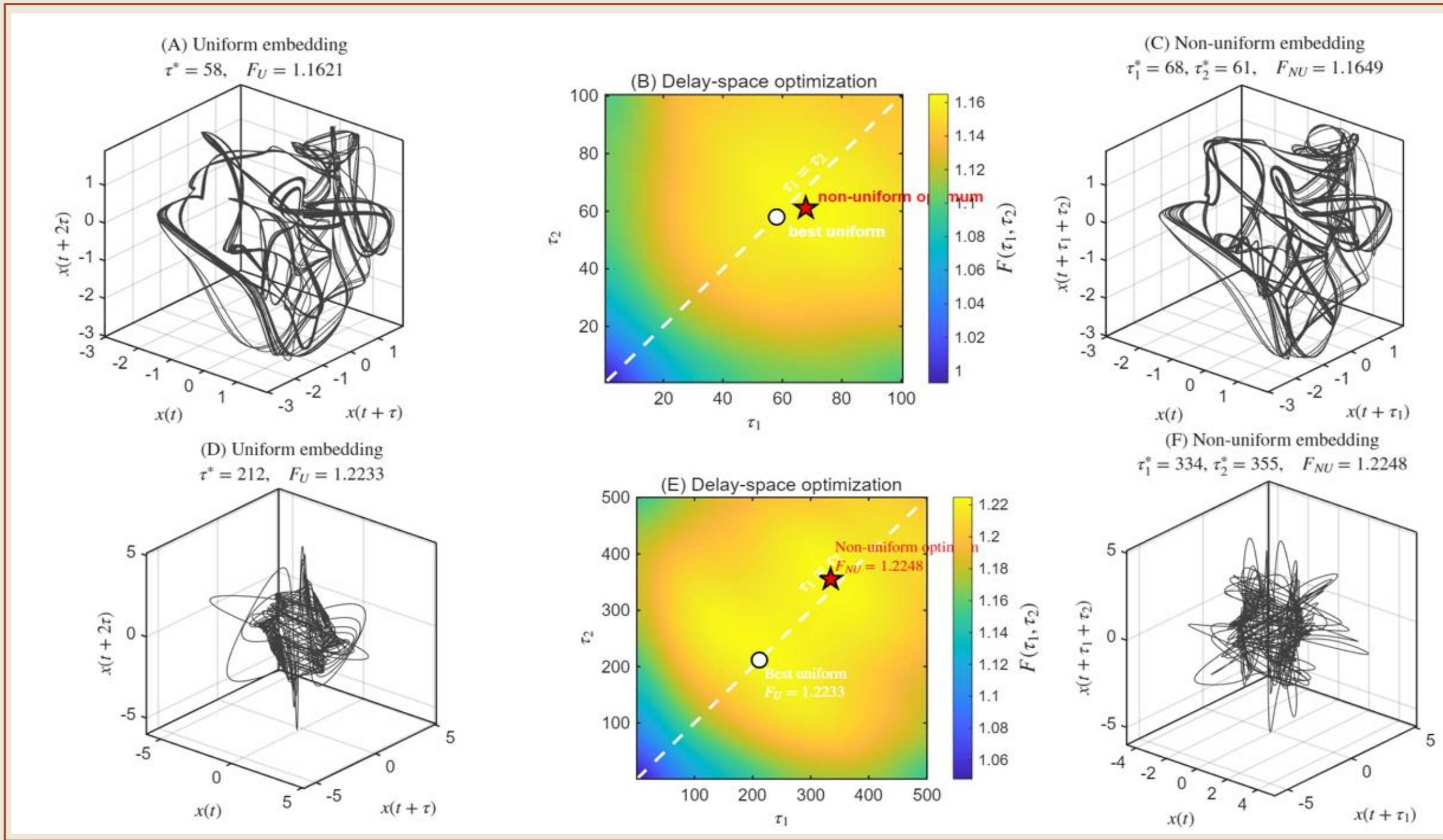


Fig 2. Geometry-based optimization of uniform and non-uniform delay embeddings for two dynamical systems: (A)-(C) Thomas system, (D)-(F) Dadras system.

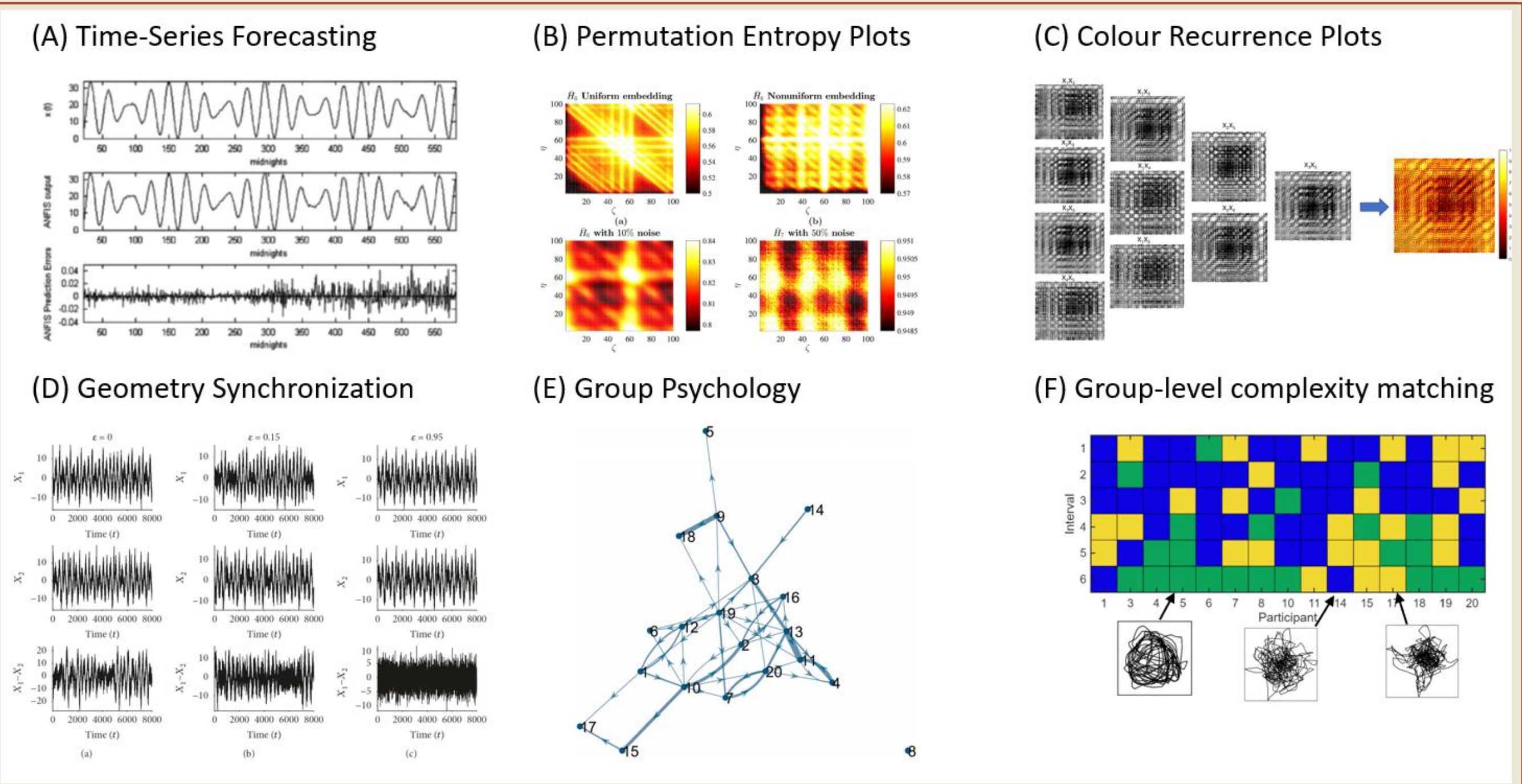


Fig 3. Representative applications of non-uniform embedding in nonlinear time-series analysis: (A) time-series forecasting [2]; (B) permutation entropy analysis [3]; (C) colour recurrence plots [4]; (D) geometry-based synchronization [1]; (E) group physiological dynamics [5]; (F) group-level complexity matching [6].

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## References

[1] I. Timofejeva, et al. "Synchronization measure based on a geometric approach to attractor embedding using finite observation windows." Complexity 2018.1 (2018): 8259496.  
[2] M. Ragulskis and K. Lukoseviciute. "Non-uniform attractor embedding for time series forecasting by fuzzy inference systems." Neurocomputing 72.10-12 (2009): 2618-2626.  
[3] M. Tao, et al. "Permutation entropy based on non-uniform embedding." Entropy 20.8 (2018): 612.  
[4] M. Landauskas, M. Cao, and M. Ragulskis. "Permutation entropy-based 2D feature extraction for bearing fault diagnosis." Nonlinear Dynamics 102.3 (2020): 1717-1731.  
[5] I. Timofejeva, et al. "Identification of a group's physiological synchronization with earth's magnetic field." International journal of environmental research and public health 14.9 (2017): 998.  
[6] Qammar Naseha Wafa, et al. "State-Space Attractor Morphology as a Marker of Complexity Matching in Group Heart Rate Variability Dynamics." Applied Sciences 16.18 (2026): 9172.