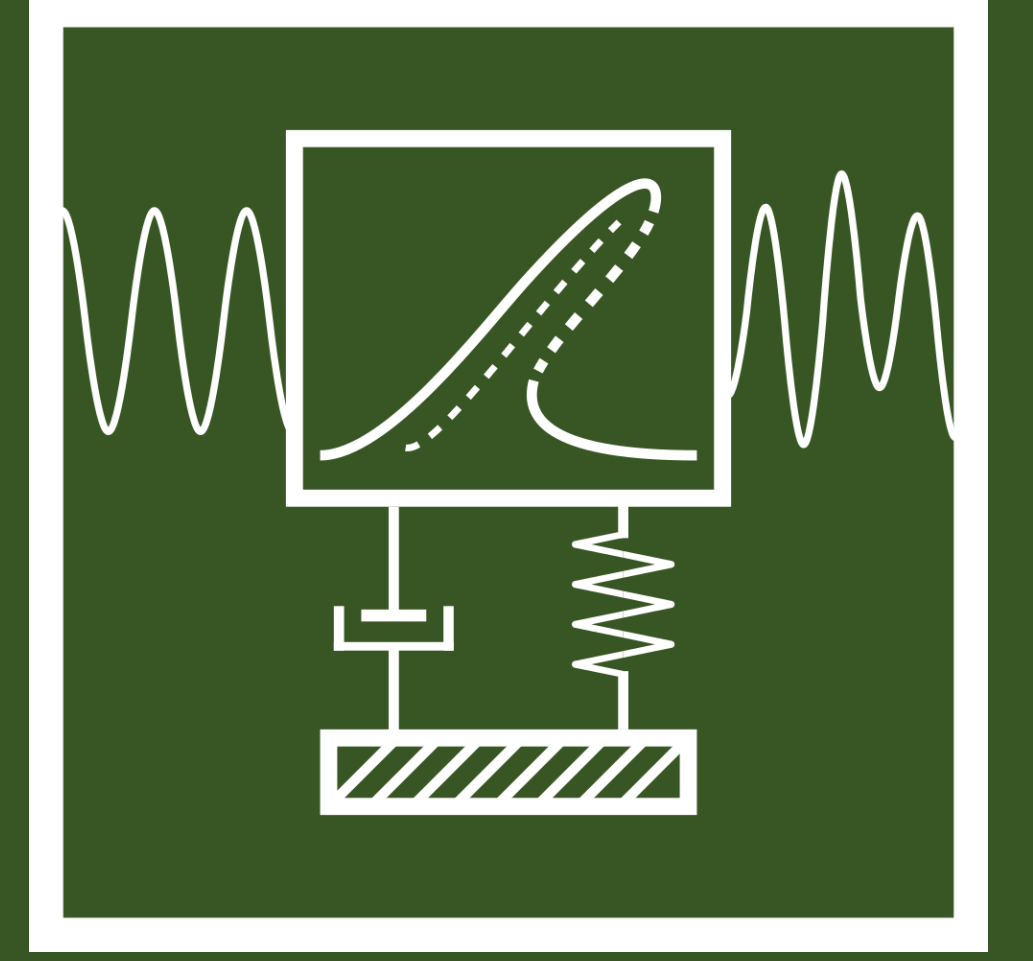


Hierarchical Arnold tongues in the Caputo fractional standard map

Rasa Smidtaite, Ugne Orinaite and Minvydas Ragulskis*

Research Group for Nonlinear Systems, Kaunas University of Technology, Studentu 50-146, Kaunas, LT-51368, Lithuania

Corresponding author e-mail: minvydas.ragulskis@ktu.lt



A Abstract

Fractional memory and nilpotent matrix algebra are combined in the Caputo fractional standard map to investigate higher-dimensional nonlinear dynamics. Their interaction produces hierarchical families of Arnold tongues associated with divergent transients – structures absent from the classical standard map and the non-fractional nilpotent formulation. H-rank analysis is used to quantify algebraic complexity and relate hierarchical level to transient growth and mode-locking behavior.

01 Methods

Fractional matrix dynamics

$$\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)} - \frac{K}{\Gamma(\alpha-1)} \left[\sum_{r=0}^{t-1} V_{\alpha}^2(t-r+1) \sin \mathbf{X}^{(r)} + \sin \mathbf{X}^{(t)} \right],$$

$$\mathbf{X}^{(t+1)} = \mathbf{X}^{(t)} + \mathbf{P}^{(0)} - \frac{K}{\Gamma(\alpha)} \sum_{r=0}^t V_{\alpha}^1(t-r+1) \sin \mathbf{X}^{(r)}, \quad V_{\alpha}^s(m) = m^{\alpha-s} - (m-1)^{\alpha-s}.$$

Caputo memory

Power-law kernel retains the full trajectory history.

Nilpotent hierarchy

$$\mathbf{X}^{(t)} = \rho_{0,x}^{(t)} \mathbf{I} + \sum_{j=1}^{n-1} \mu_{j,x}^{(t)} \mathbf{N}_j, \quad \mathbf{N}_i \mathbf{N}_j = \begin{cases} \mathbf{N}_{i+j}, & i+j \leq n-1, \\ \mathbf{0}, & i+j \geq n. \end{cases}$$

Nilpotent algebra

Matrix coefficients μ_j become coupled through higher-order terms.

$$\mu_{j,x}^{(t+1)} = \mu_{j,x}^{(t)} + \mu_{j,p}^{(0)} - \frac{K}{\Gamma(\alpha)} \sum_{r=0}^t V_{\alpha}^1(t-r+1) \sum_{m=1}^j \frac{\sin(\rho_{0,x}^{(r)} + \frac{m\pi}{2})}{m!} \sum_{i_1+\dots+i_m=j} \prod_{\ell=1}^m \mu_{i_{\ell},x}^{(r)}.$$

Algebraic-complexity measure

$$\text{Hrank}_{\epsilon}(y) = \# \{k : \sigma_k(\mathbf{H}_h(y)) > \epsilon\}.$$

H-rank mapping

Hankel singular values quantify algebraic sequence complexity.

02 Results

- Arnold-type tongue structures are present in all three hierarchical components $\mu_{1,x}$, $\mu_{2,x}$ and $\mu_{3,x}$.
- The average algebraic complexity increases with hierarchical order, accompanied by progressively finer H-rank structure.
- A single parameter point may occupy different H-rank regions at different hierarchical levels, showing that the layer index j is dynamically significant.
- The auxiliary variables diverge algebraically rather than exponentially: higher levels follow progressively stronger power-law amplification.

Growth hierarchy

$$|\mu_{j,x}^{(t)}| \sim t^{\beta}, \quad \beta > 0$$

Polynomial growth explains why classical Lyapunov exponents do not capture the effective amplification of the hierarchy.

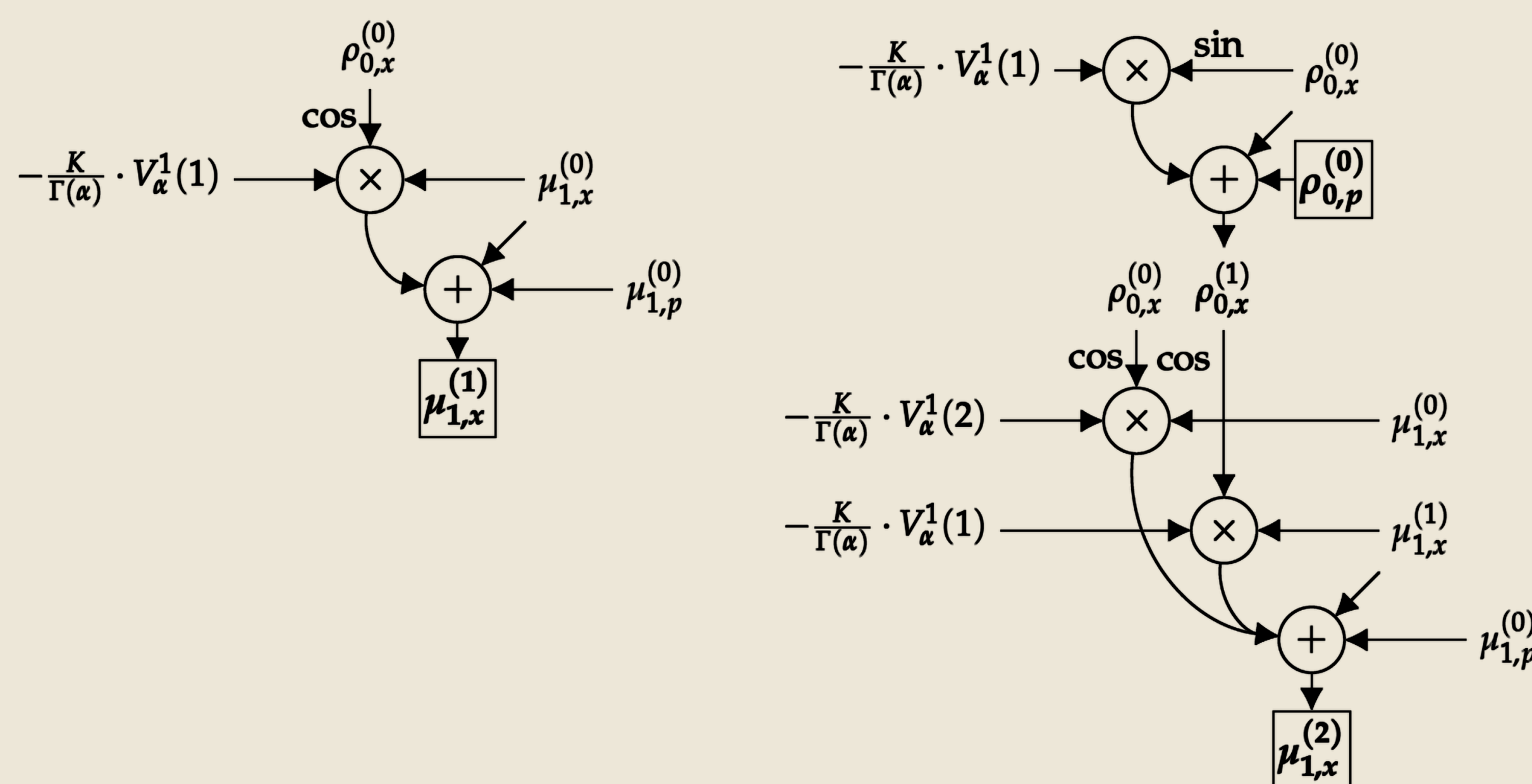


Fig. 1. Schematic construction of $\mu_{1,x}^{(t+1)}$ for $t = 0, 1$, combining cosine-modulated multiplicative terms with contributions from preceding levels. Analogous schemes apply to $\mu_{2,x}^{(t+1)}$ and $\mu_{3,x}^{(t+1)}$.

03 Discussion

- Fractional memory alone and nilpotent algebra alone do not generate the observed hierarchy; the new structures arise from their interaction.
- For idempotent matrices, the fractional map retains the dynamical complexity of the scalar counterpart; nilpotent coupling produces the richer regime.
- Low H-rank regions correspond to mode-locked dynamics, while high H-rank regions are associated with quasiperiodic or more complex trajectories.

04 Conclusions

- High-dimensional nilpotent matrices generate hierarchical Arnold tongues of divergence.
- The hierarchical index j controls transient algebraic complexity and amplification.
- The hierarchy provides an additional degree of freedom for information encoding and related applications.

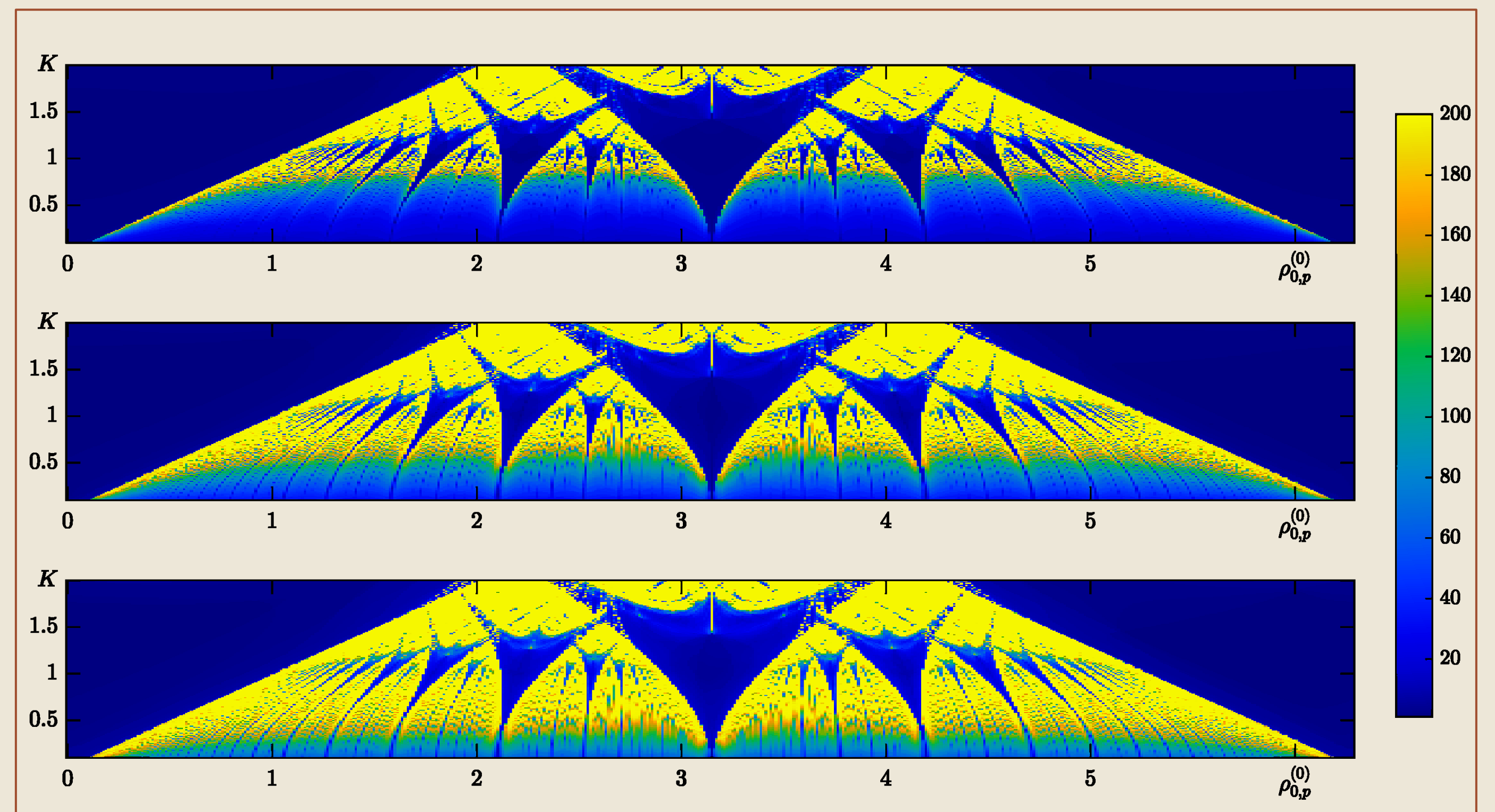


Fig. 2. H-rank maps for $\mu_{1,x}$ (top), $\mu_{2,x}$ (middle) and $\mu_{3,x}$ (bottom). Increasing yellow dominance indicates greater algebraic complexity at higher hierarchical levels.

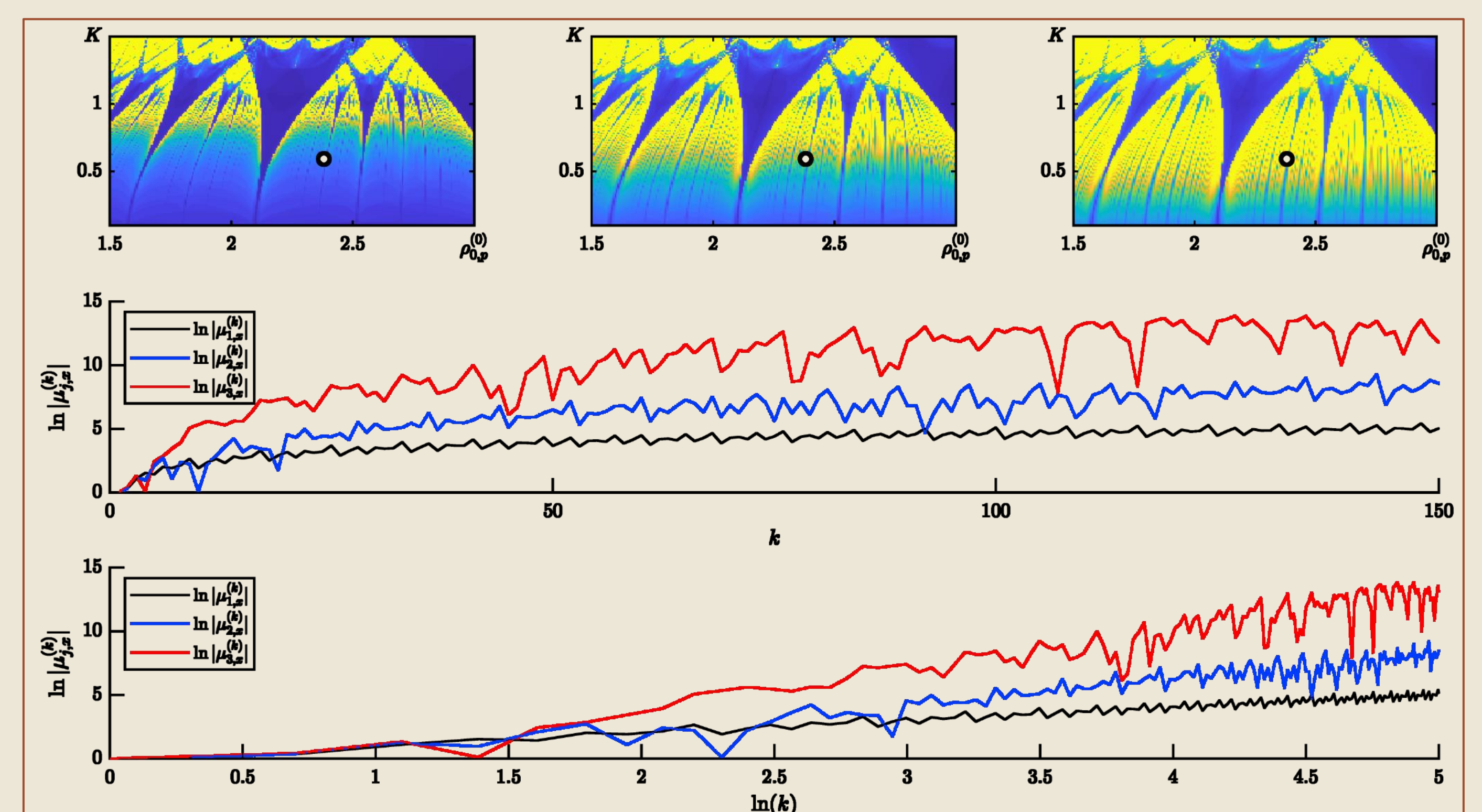


Fig. 3. The same parameter point ($\rho_{0,p} = 2.3813$, $K = 0.5926$) occupies different H-rank regions across the hierarchy and shows progressively stronger transient growth.

Key implication: hierarchical layer j is an additional structural parameter for transient complexity and may be exploited in information-encoding schemes.

Acknowledgements

Funded by the Research Council of Lithuania (LMTLT), Project No. S-MIP-25-52 — “Finite-time divergence in differential equations of nilpotent matrices — theory and applications (DENM)”.

References

- [1] U. Orinaite et al., “How Does the Fractional Derivative Change the Complexity of the Caputo Standard Fractional Map,” *Int. J. Bifurcation Chaos*, 34, 2450085, 2024.
- [2] U. Orinaite, R. Smidtaite, M. Ragulskis, “Arnold tongues of divergence in the Caputo fractional standard map of nilpotent matrices,” *Nonlinear Anal. Model. Control*, 30, 1–18, 2025.
- [3] R. Smidtaite, Z. Navickas, M. Ragulskis, “Clocking divergence of iterative maps of matrices,” *Commun. Nonlinear Sci. Numer. Simul.*, 95, 105589, 2021.
- [4] M. Ragulskis, Z. Navickas, “The rank of a sequence as an indicator of chaos in discrete nonlinear dynamical systems,” *Commun. Nonlinear Sci. Numer. Simul.*, 16, 2894–2906, 2011.