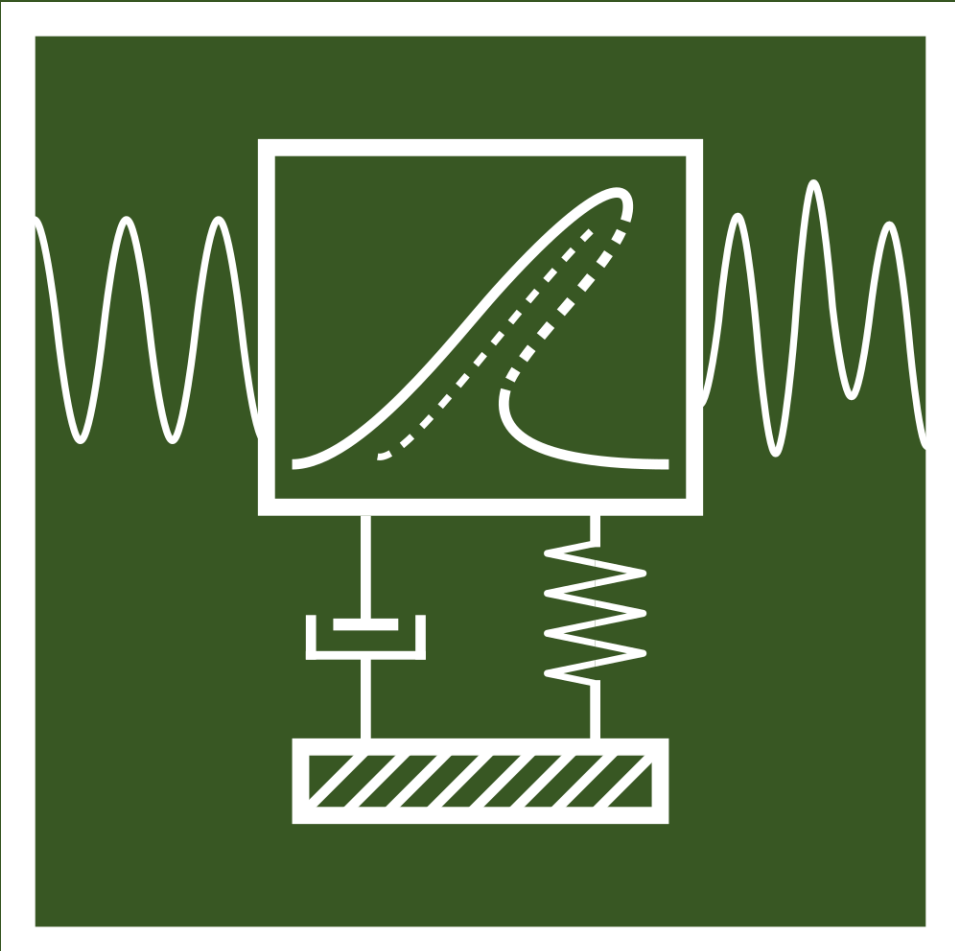


Mapping Arnold Tongues of Divergence in Fractional Matrix Dynamics

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A Abstract

Replacing the scalar variables of the Caputo fractional standard map with nilpotent matrices produces transients that grow, decay, or settle into periodic motion. The H-rank algorithm cannot tell these apart — equal algebraic complexity can hide opposite long-term fates. Combining PCA with k-means clustering instead classifies each transient directly, revealing a rich family of Arnold tongues of divergence.

01 Methods

Scalar variables in the Caputo fractional standard map are replaced by 2×2 matrices sharing a common nilpotent N , so $X = \lambda I + \mu N$. The map stays closed under iteration and reduces to two intertwined scalar recursions for $\lambda^{(k)}$ and $\mu^{(k)}$. We sweep initial phase $\lambda_p^{(0)} \in [0, 2\pi]$ and kick strength $K \in [0, 2]$ at $\alpha = 1.001$, generating 1000-step transients of $\mu_x^{(k)}$.

02 Results

H-ranks for the scalar and nilpotent-matrix maps produce almost identical Arnold-tongue ornaments (Fig. 3) — even though nilpotent matrices are known to cause finite-time or explosive divergence. This is deceptive: H-rank measures algebraic complexity, not convergence. Two transients with equal H-rank can diverge, decay, or settle into oscillation. Trajectories of $\mu_x(k)$ also run far longer than the classical 399-sample observation window suggests, yet double-precision arithmetic stays stable throughout. PCA followed by k-means clustering on these transients instead reveals 5–30 distinct classes (Fig. 2):

- dark blue — quick convergence to a quiet state
- yellow — complex, oscillating divergence
- orange sub-tongues — transient approach to asymptotically periodic motion
- green — monotonous, linearly diverging trajectories

Each tongue is a slightly different transient “sound” — like a different piano key.

03 Discussion

Nilpotent matrices are known to trigger finite-time or explosive divergence, and periodic orbits generally don't exist in fractional-order systems — yet the Arnold-tongue ornament survives almost unchanged (Fig. 3). The nilpotent-matrix map alone, without fractional memory, produces no tongues at all: it is specifically the combination of Caputo memory and nilpotent divergence that creates this new structure.

04 Conclusions

We introduce Arnold tongues of divergence — regions that classify not just resonance but the type of divergent transient (growth, decay, oscillation, asymptotic periodicity). H-rank alone cannot resolve them; PCA + k-means clustering can. Future work: coupled nilpotent hypermaps, other fractional derivatives (Riemann–Liouville, Hadamard, Hilfer), and applications in secure information encoding.

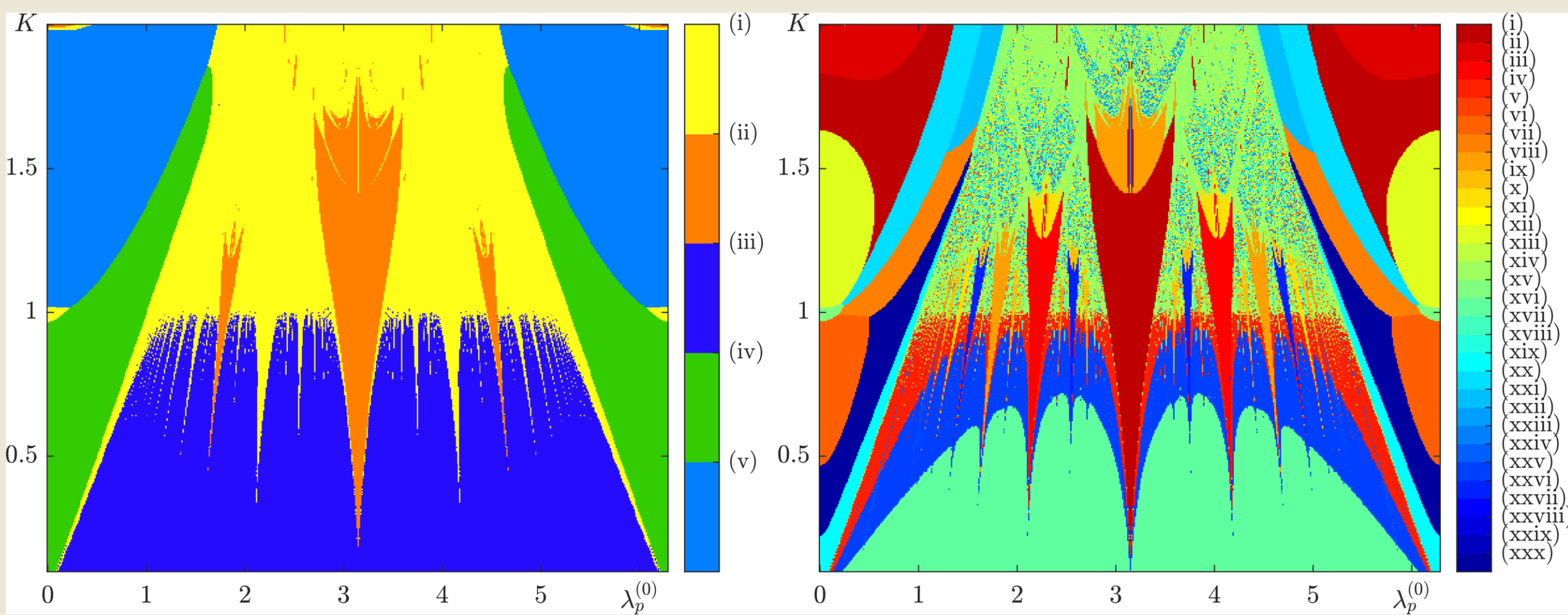


Fig 2. Arnold tongues of divergence, classified by PCA + k-means clustering of the transients of $\mu_x^{(k)}$: 5 classes (left) and 30 classes (right).

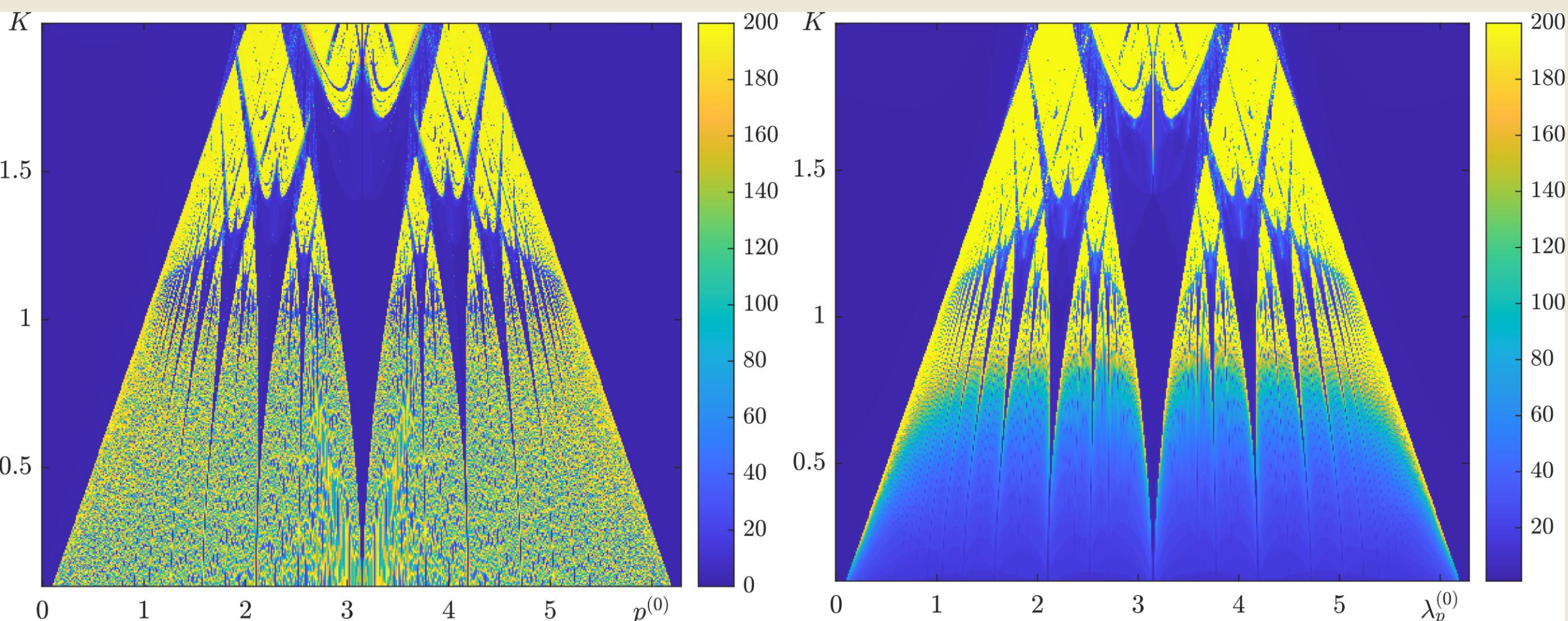


Fig 3. H-rank patterns look almost identical for the scalar map (left) and the nilpotent-matrix map (right) — yet the underlying transients diverge completely.

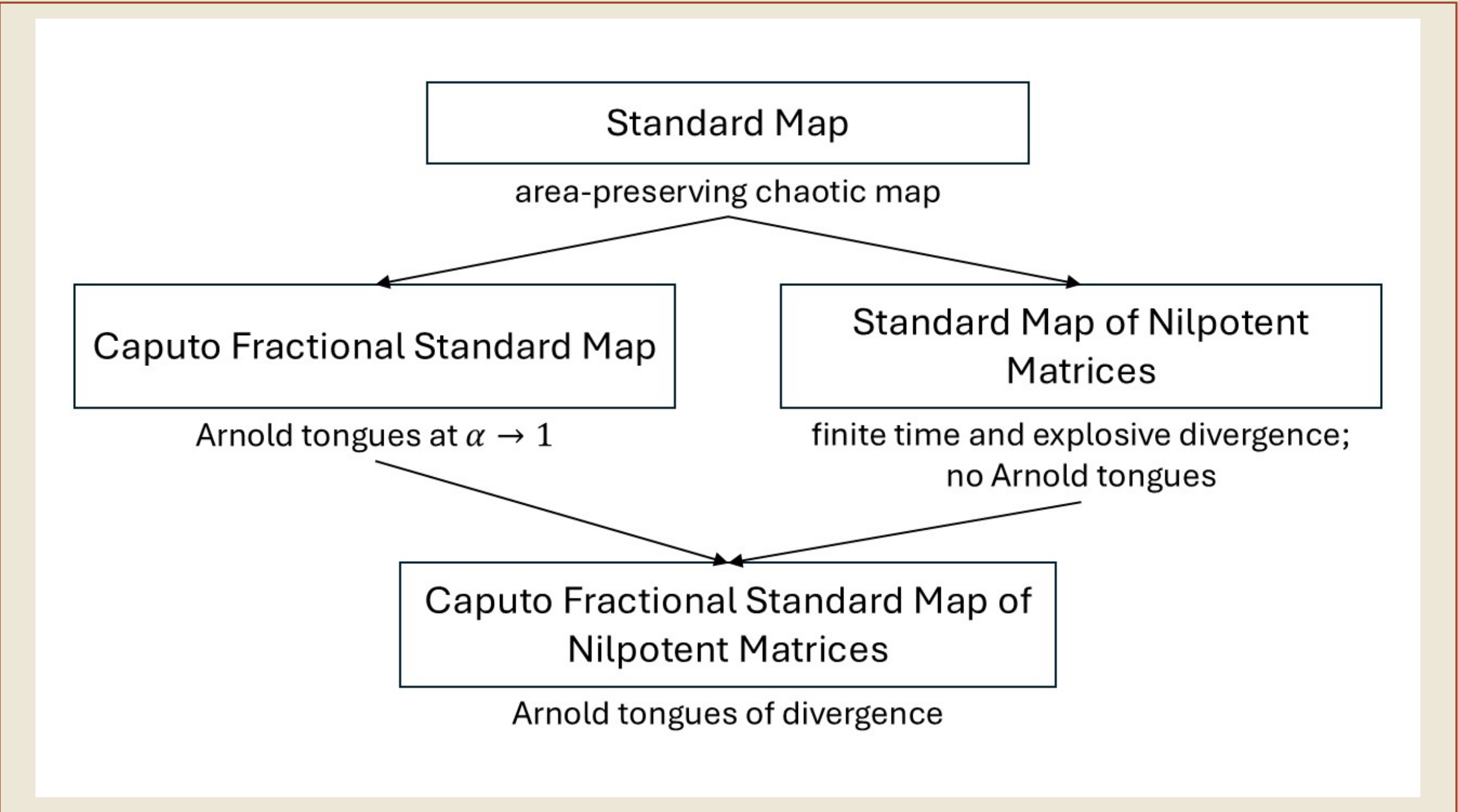


Fig 1. How the models relate: fractional memory and nilpotent-matrix divergence combine to produce Arnold tongues of divergence.

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References

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